1. $$\frac{1}{(z-1)(z-2)}$$

$$= \frac{(-1)}{z-1} + \frac{1}{z-2}$$

a) For $1<|z|<2$,

$$\frac{-1}{z-1} = \frac{1}{z}\,\frac{1}{1-\left(\frac{1}{z}\right)} \qquad \left|\frac{1}{z}\right| < 1$$

$$= \frac{1}{z}\cdot\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n$$

$$= \sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^{n+1}$$

$$\frac{1}{z-2} = \frac{-1}{2}\cdot\frac{1}{1-\left(\frac{z}{2}\right)} \qquad \left|\frac{z}{2}\right| < 1$$

$$= -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$$

$$\therefore \frac{1}{(z-1)(z-2)} = -\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^{n+1} - \sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}$$

b) For $1<|z-3|<2$

$$\frac{1}{z-1} = \frac{1}{(z-3)+2}$$

$$= \frac{1}{2}\cdot\frac{1}{1+\left(\frac{z-3}{2}\right)} \qquad \left|\frac{z-3}{2}\right| < 1$$

$$= \frac{1}{2}\cdot\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-3}{2}\right)^n$$

$$\frac{1}{z-2} = \frac{1}{(z-3)+1}$$

$$= \frac{1}{z-3}\cdot\frac{1}{1+\frac{1}{z-3}} \qquad \left|\frac{1}{z-3}\right| < 1$$

$$= \frac{1}{z-3}\cdot\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{z-3}\right)^n$$

$$\therefore \frac{1}{(z-1)(z-2)} = \sum_{n=0}^{\infty}(-1)^{n+1}\frac{(z-3)^n}{2^{n+1}} + \sum_{n=0}^{\infty}(-1)^n\frac{1}{(z-3)^{n+1}}$$

2(a) $\frac{1}{z+z^2} = \frac{1}{z}\left(\frac{1}{1+z}\right)$

$= \frac{1}{z} \sum_{n=0}^{\infty} (-z)^n$

$= \frac{1}{z} - 1 + z - \cdots$

$\therefore \operatorname{Res}_{z=0} \frac{1}{z+z^2} = 1$

(b) $z\cos\left(\frac{1}{z}\right) = z \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(\frac{1}{z}\right)^n$

$= z - \frac{1}{2!}\left(\frac{1}{z}\right) + \frac{1}{4!}\left(\frac{1}{z}\right)^3 - \cdots$

$\therefore \operatorname{Res}_{z=0} z\cos\left(\frac{1}{z}\right) = -\frac{1}{2}$

(c) $\frac{\cot z}{z^4} = \frac{1}{z^4}\left(\frac{1}{z} - \frac{z}{3} - \frac{z^3}{45} - \cdots\right)$

$= \frac{1}{z^5} - \frac{1}{3z^3} - \frac{1}{45z} - \cdots$

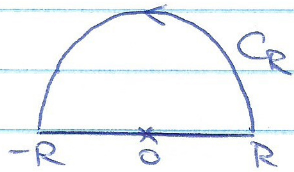
$\therefore \operatorname{Res}_{z=0} \frac{\cot z}{z^4} = -\frac{1}{45}$

(d) $\frac{z^3+2z+1}{z^2(z+1)} = \frac{f(z)}{z^2}$ where $f(z) = \frac{z^3+2z+1}{z+1}$

$\operatorname{Res}_{z=0} \frac{z^3+2z+1}{z^2(z+1)} = f'(0) = 1$

3(a) Let $f(z) = \dfrac{z^2}{(z^2+1)(z^2+4)}$

$$\int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz = 2\pi i\left(\operatorname*{Res}_{z=i} f(z) + \operatorname*{Res}_{z=2i} f(z)\right)$$

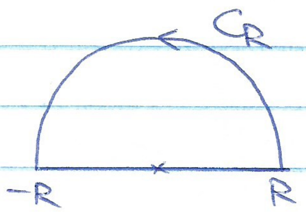


- $\operatorname*{Res}_{z=i} f(z) = -\dfrac{1}{6i}$ $\quad \operatorname*{Res}_{z=2i} f(z) = \dfrac{1}{3i}$
- $\left|\int_{C_R} f(z)\,dz\right| \le 2\pi R \dfrac{R^2}{(R^2-1)(R^2-4)}$

$$= \frac{2\pi}{R - \frac{5}{R} + \frac{4}{R^3}}$$ which tends to zero as R tends to $+\infty$.

$$\therefore \int_0^{+\infty} \frac{x^2}{(x^2+1)(x^2+4)}\,dx = \frac{\pi}{6}$$

(b) Let $f(z) = \dfrac{z}{z^2+3}$



$$\int_{-R}^{R} f(x)e^{2ix}\,dx + \int_{C_R} f(z)e^{2iz}\,dz$$

$$= 2\pi i \operatorname*{Res}_{z=\sqrt{3}i} \left(f(z)e^{2iz}\right)$$

- $\operatorname*{Res}_{z=\sqrt{3}i} f(z)e^{2iz} = \frac{1}{2}e^{-2\sqrt{3}}$
- $\int_{C_R} f(z)e^{2iz}\,dz$ tends to 0 as R tends to $+\infty$

by Jordan's lemma. $\left(\because |f(z)| \le \dfrac{R}{R^2-3}\right)$

$$\therefore \int_{-\infty}^{+\infty} \frac{x\sin 2x}{x^2+3}\,dx = 2\pi \cdot \frac{1}{2}e^{-2\sqrt{3}}$$

$$\int_0^{+\infty} \frac{x\sin 2x}{x^2+3}\,dx = \frac{\pi}{2e^{2\sqrt{3}}}$$

4 Since z_0 is a zero of $f(z)$ and $g(z)$ of order $m>0$.

$f(z) = (z-z_0)^m \tilde{f}(z)$ and $g(z) = (z-z_0)^m \tilde{g}(z)$

where $\tilde{f}$ and $\tilde{g}$ are analytic and nonzero at $z=z_0$.

Furthermore, $\tilde{f}(z_0) = \dfrac{f^{(m)}(z_0)}{m!}$ and $\tilde{g}(z_0) = \dfrac{g^{(m)}(z_0)}{m!}$

Therefore,

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \lim_{z \to z_0} \frac{\tilde{f}(z)}{\tilde{g}(z)}$$

$$= \frac{\tilde{f}(z_0)}{\tilde{g}(z_0)}$$

$$= \frac{f^{(m)}(z_0)}{g^{(m)}(z_0)}$$